Geometry		Repr. theory
$\boxed{X}$	$\longrightarrow$	$\boxed{\mathfrak{g} \hookrightarrow \mathrm{End}(\mathbb{C}^n)}$
$H^*(X)$		• comult. Δ
$X_1 \times X_2 = X^T$		$\mathfrak{g} \longrightarrow 1 \otimes \mathfrak{g} + \mathfrak{g} \otimes 1$
$H^*(X_1) \otimes H^*(X_2)$		$\Delta = (12)\Delta$
$\cong H^*(X_2) \otimes H^*(X_1)$		•• V_1, V_2 repr. $V_1 \otimes V_2 \cong V_2 \otimes V_1$
$\big\downarrow$ equiv. para.		$\big\downarrow$ loop space
$\boxed{A \circlearrowright X}$	$\longrightarrow$	$\boxed{\mathfrak{g}[t] \hookrightarrow \mathrm{End}(\mathbb{C}^n(t))}$
$H^*_A(X)$		• $\mathfrak{g}[t] \longrightarrow 1 \otimes \mathfrak{g}[t] + \mathfrak{g}[t] \otimes 1$
$X_1 \times X_2 = X^T$		•• $V_1(t_1), V_2(t_2)$ repr.
$H^*_A(X_1) \otimes H^*_A(X_2)$		$V_1(t_1) \otimes V_2(t_2) \cong V_2(t_2) \otimes V_1(t_1)$
$\big\downarrow \times \mathbb{C}^\times_\hbar$		$\big\downarrow$ Hopf algebra deformation
$\boxed{A \times \mathbb{C}^\times_\hbar \circlearrowright X}$	$\longrightarrow$	$\boxed{\text{Yangian}}$
$H^*_{A \times \mathbb{C}^\times_\hbar}(X)$		• comult. $\widetilde{\Delta}$
$H^*_{A \times \mathbb{C}^\times_\hbar}(X_1) \otimes H^*_{A \times \mathbb{C}^\times_\hbar}(X_2$		$\widetilde{\Delta} \neq (12)\widetilde{\Delta}$
$H^*_{A \times \mathbb{C}^\times_\hbar}(X_1) \otimes H^*_{A \times \mathbb{C}^\times_\hbar}(X_2)$ geometric R-matrix	$\longrightarrow$	•• $V_1(t) \otimes V_2(s) \xrightarrow{\text{R-matrix}} V_2(s) \otimes V_1(t)$
$H^*_{A \times \mathbb{C}^\times_\hbar}(X^T) \longrightarrow H^*_{A \times \mathbb{C}^\times_\hbar}(X)$ Stab. envelope	$\longrightarrow$	$R = \text{lower} \circ \text{upper}$

Attracting set.

$T \curvearrowright (X, \omega)$ holo. Sympl. mfd. with Hamiltonian T-action.

X^T = fixed pts set of X w.r.t. T action

for $\forall \xi \in \mathrm{Cochar}(T)$, Z a connected component of X, we say $x \in \mathrm{Attr}_\xi(Z)$ if.

$\lim_{t \to 0} \xi(t) x \ \exists$.

Partial order

Relation: $Z' \lessapprox Z$ if $Z' \cap \overline{\mathrm{Attr}(Z)} \neq \phi$

Claim: Relation $\lessapprox$ is partial order.

Thm: define $\mathrm{Attr}^f(Z) = \bigsqcup_{Z' \lessapprox Z} \mathrm{Attr}(Z')$, then $\mathrm{Attr}^f(Z)$ is closed if X is sympl. resolution.

Pf: Choose $X \xrightarrow{\pi} X_0 \overset{i}{\hookrightarrow} V$ A-equiv. & proper (sympl. res.)

$V_{\geq 0}$ subspace with non-negative A-weight.

means closure of Attr still Attr

$\Rightarrow \overline{\mathrm{Attr}(Z)} \longrightarrow V_{\geq 0}$ $\Rightarrow x \in \overline{\mathrm{Attr}(Z)} \setminus \mathrm{Attr}(Z) \longrightarrow V_{\geq 0} \cap X_0$

$\Rightarrow \lim_{t \to 0} \pi(\xi(t) x) \ \exists \Rightarrow \lim_{t \to 0} \xi(t) x \in \overline{\mathrm{Attr}(Z)} \cap X^A \ \exists$ by properness

Equivariant roots & Chamber structure.

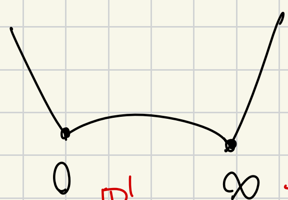
$T \curvearrowright (X, \omega)$ holo. sympl. mfd with Hamilitonian T-action, X^T the fixed pts set.

we say σ is a equivariant root

if σ = some weights of T action on normal bundle of X^T, i.e. taking value in $H^*_T(pt) \cong \mathfrak{t}^*$ (Then define a Chamber structure on $\mathfrak{t}$)

Ex.

$X = T^*\mathbb{P}^1$



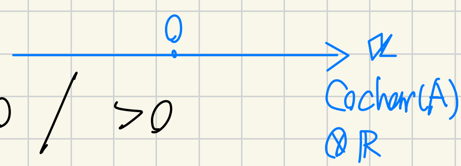
$T = \mathbb{C}^\times_a \curvearrowright T^*\mathbb{C}^2/\!/\mathbb{C}^\times \quad (\underbrace{x_1, x_2}_{\mathbb{P}^1}, \underbrace{y_1, y_2}_{\text{fiber}}) \longmapsto (x_1, a x_2, y_1, a_2^{-1} y_2)$

$T_0\mathbb{P}^1 = \{(1, x), x \in \mathbb{C}\}$

$\mathbb{C}^\times_a \curvearrowright T_{0/\infty}\mathbb{P}^1 : (1, x) \longmapsto (1, ax) / (1, a^{-1}x)$

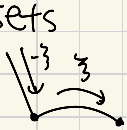
$\Rightarrow (\mathbb{C}^\times)^2_{a_1, a_2} \curvearrowright T^*_{0/\infty}\mathbb{P}^1 : (1, x) \longmapsto (1, a^{-1}x) / (1, ax)$

$\Rightarrow$ equiv. root : a & $-a$



pick $\xi \in \mathrm{Cochar}\, T$ s.t. $\langle \xi, a \rangle < 0 \; / \; > 0$

$\Rightarrow$ Attr. sets



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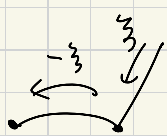


$\Rightarrow$ change chamber, change Attr. sets.

Def. of Stable envelopes.

Choose a Chamber C & $\forall$ cochar. $\xi|_C > 0$

Stable envelope is a $H^*_{A\times\mathbb{C}^\times_\hbar}(pt)$ - map

$$H^*_{A\times\mathbb{C}^\times_\hbar}(X^T) \longrightarrow H^*_{A\times\mathbb{C}^\times_\hbar}(X)$$

s.t.

(i) $\forall Z \subseteq X^T$ a connected component

$$\mathrm{Supp}(\mathrm{Stab}_\xi(Z)) \subseteq \mathrm{Attr}^f_\xi(Z)$$

(ii) $\mathrm{stab}_\xi(Z)\big|_Z = e(N_-)$ (diag)

(iii) $\deg_A \mathrm{stab}_\xi(Z)\big|_{Z'}, \; Z' < Z \quad < \frac{1}{2}\,\mathrm{codim}\, Z'$ (off diag)

Then we shall see

(1) Stab are upper tri. (partial order $\Rightarrow$ full order for Nak. quiver varieties)

(2) Why Stab give R-metrices.

(3) if X is sympl. resolution, when $\hbar \to 0$
Stab is diagonal matrix.

$\boxed{\text{For (1)}}$ We introduce a finer order relation

Pick $\forall$ A-linearized ample line bundle L over X,

then if Z_1 & Z_2 are connected by curve $\overline{\mathfrak{z}(t)} = \mathbb{P}^1$

we have

$$0 < \int_{\mathbb{P}^1} c_1(L) = \frac{L|_0}{e(T_0\mathbb{P}^1)} + \frac{L|_\infty}{e(T_\infty\mathbb{P}^1)}$$

$$= \frac{1}{\mathfrak{z}}(L|_0 - L|_\infty)\Big|_C \quad (\mathfrak{z}|_C > 0)$$

Note that LHS doesn't depends on the choice of linearization,

Then we say $Z_1 < Z_2$ if $L|_{Z_1} < L|_{Z_2}$

In particular, when X is Nak. quiver varieties.

$\mathcal{M}_\theta(v, w)$, we pick the ample line bundle

$\sum \theta_i c_1(V_i)$

At fixed component $Z_\eta = \mathcal{M}_\theta(v-\eta, w_1) \times \mathcal{M}_\theta(\eta, w_2)$

we have

$$wt\left(\sum \theta_i c_1(V_i)\right)\Big|_{Z_\eta} = \sum \theta_i \eta_i$$

Then we say $Z_\eta > Z_{\eta'}$ if $\sum \theta_i \eta_i > \sum \theta_i \eta_i'$

For generic θ_i we have strict full order

For 2

The main tool is uniqueness of stable envelopes.

Thm.

$\forall\ \gamma \in H_T^*(X)$ s.t. $\operatorname{supp} \gamma \subseteq \operatorname{Attr}^f(Z)$ for some Z

$\deg_A \gamma|_{Z'} < \frac{1}{2}\operatorname{codim} Z'$, $\forall\ Z' \leq Z$

$\Rightarrow \gamma = 0$

Pf:

By definition we know $\gamma \in H^*(X, X \setminus \operatorname{Attr}^f(Z)) = H_*^{BM}(\operatorname{Attr}^f(Z))$

$\forall\ X \hookrightarrow M$ closed embedding for M smth, we have following diagram commutes.

$$\begin{tikzcd}
\gamma \in H^*(X, X\setminus \operatorname{Attr}^f(Z)) \arrow[r] \arrow[d, "U[1] \in H^*(M, M\setminus X)"] & {[\gamma]_{BM}} \in H^*_{BM}(\operatorname{Attr}^f(Z)) \arrow[d, "i_*"] \\
\gamma U[1]\ H^*(M, M\setminus X) \arrow[r] & i_*[\gamma]_{BM} \in H^*_{BM}(X)
\end{tikzcd}$$

$\Rightarrow \quad \gamma \cap [X]_{BM} = i_*[\gamma]_{BM}$

Since we have $Z \xrightarrow{f_1} \operatorname{Attr}(Z) \xrightarrow{f_2} \operatorname{Attr}^f(Z) \xrightarrow{i} X$

$\Rightarrow (i \circ f_2 \circ f_1)^* \gamma \cap [Z]_{BM} = f_1^* \circ f_2^*\, i^* i_* [\gamma]_{BM}$

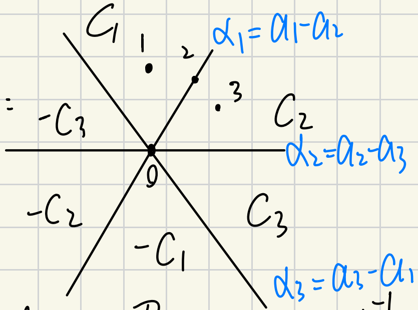
$= e(N_Z^-)\, f_1^* \circ f_2^* [\gamma]_{BM}$

Since $\deg e(N_Z^-) = \frac{1}{2}\operatorname{codim} Z^- \Rightarrow f_2^*[\gamma]_{BM} = 0 \Rightarrow \operatorname{supp} \gamma \subseteq \operatorname{Attr}^f(Z')$ $Z' < Z$, by induction $\square$

Consider Stab : $H^*_{A\times\mathbb{C}^\times_\hbar}(X^T) \longrightarrow H^*_{A\times\mathbb{C}^\times_\hbar}(X^{T'})$

s.t. $T' \subseteq T$ & Ker: $t^* \longrightarrow t'^*$ = some equivariant root α.

chamber structure:



1, 3 stand for generic cochar of T & 2 stand for generic cochar of T'

$\Rightarrow$ We define $R_{C_1,C_2} = \mathrm{Stab}^{-1}_{3\to 2}\circ \mathrm{Stab}_{1\to 2} \in \mathrm{End}(H^*_{A\times\mathbb{C}^\times_\hbar}(X^T))\otimes \mathbb{Q}(t)$

$$R_{C_1,C_2} \,\|\, R_{C_1,C_2}(\alpha_1) = R_{C_1,C_2}(\alpha_1-\alpha_2)$$

To define Stab^{-1} we need localizat. thm

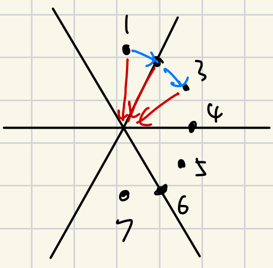
Lemma:

$$\mathrm{Stab}_{2\to 0}\circ \mathrm{Stab}_{1\to 2} = \mathrm{Stab}_{1\to 0}$$

Pf: immediately follows from the fact that Stab is upper tri. & unique

Thm:

$$R_{C_3,C_1}(a_3-a_1)\circ R_{C_2,C_3}(a_2-a_3)\circ R_{C_1,C_2}(a_1-a_2) = R_{C_2,C_1}(a_2-a_1)\circ R_{C_3,C_2}(a_3-a_2)\circ R_{C_1,C_3}(a_1-a_3)$$



$$\mathrm{Stab}^{-1}_{3\to 2}\circ \mathrm{Stab}_{1\to 2} = \mathrm{Stab}^{-1}_{3\to 2}\circ \mathrm{Stab}^{-1}_{2\to 0}\circ \mathrm{Stab}_{1\to 0}$$
$$= \mathrm{Stab}^{-1}_{3\to 0}\circ \mathrm{Stab}_{1\to 0}$$

$$\Rightarrow R_{C_3,C_1}\circ R_{C_2,C_3}\circ R_{C_1,C_2} = \mathrm{Stab}^{-1}_{7\to 0}\circ \mathrm{Stab}_{1\to 0}$$
$$= R_{C_2,C_1}\circ R_{C_3,C_2}\circ R_{C_1,C_3}$$ (Y-B equation)

$\boxed{\text{For (3)}}$ we'll show first the geometric R-matrices will degenerate to (12) without the presence of $\hbar$ by example. (True when X is conical sympl. resolution)

GKM description of $H^*_{(\mathbb{C}^\times)^3}(T^*\mathbb{P}^2)$

$$(\mathbb{C}^\times)^3_{a_1,a_2,a_3} \curvearrowright T^*_{y_1,y_2,y_3}\mathbb{C}^3_{x_1,x_2,x_3}/\!/\mathbb{C}^\times \text{ by}$$

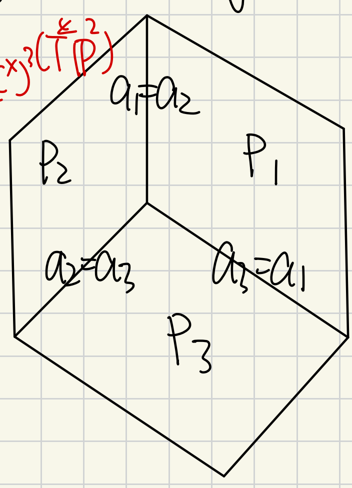
$$(x_1,x_2,x_3,y_1,y_2,y_3) \longrightarrow (a_1x_1,a_2x_2,a_3x_3,a_1^{-1}y_1,a_2^{-1}y_2,a_3^{-1}y_3)$$

we have

$$H^*_{(\mathbb{C}^\times)^3}(T^*\mathbb{P}^2) \hookrightarrow \bigoplus_{i=1}^{3} H^*_{(\mathbb{C}^\times)^3}(P_i)$$

$\{P_i\}$ the torus fixed pts

$\mathrm{Spec}\, H^*_{(\mathbb{C}^\times)^3}(T^*\mathbb{P}^2)$



we see $(\star)$

$$(e(N^-_{P_1})=(a_1-a_2)(a_1-a_3),\ 0,\ 0)$$

$$(0,\ e(N^-_{P_2})=(a_2-a_1)(a_2-a_3),\ 0)$$

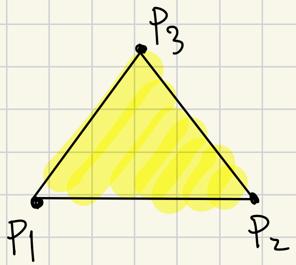
$$(0,\ 0,\ e(N^-_{P_3})=(a_3-a_1)(a_3-a_2))$$

define functions on $\mathrm{Spec}\, H^*_{(\mathbb{C}^\times)^3}(T^*\mathbb{P}^2)$

We only need to check support condition.

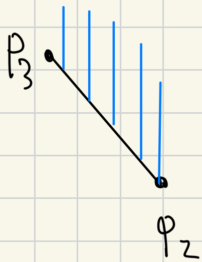
$\mathcal{C}: \ a_1 > a_2 > a_3$

L_1



$$L_1|_{P_1} = T^*_{P_1}\mathbb{P}^2 = (a_2-a_1)(a_3-a_1)$$
$$L_1|_{P_2} = T^*_{P_2}\mathbb{P}^2 = (a_1-a_2)(a_3-a_2)$$
$$L_1|_{P_3} = T^*_{P_3}\mathbb{P}^2 = (a_1-a_3)(a_2-a_3)$$

L_2



$$L_2|_{P_1} = 0$$
$$L_2|_{P_2} = (a_3-a_2)(a_2-a_1)$$
$$L_3|_{P_3} = (a_2-a_3)(a_3-a_1)$$

$\Rightarrow [L_1]+[L_2]$ satisfied $(\star)$ & $H^*_T(P_1) \xrightarrow{[L_1]+[L_2]} H^*_T(T^*\mathbb{P}^2) \xrightarrow{([L_1]+[L_2])^{-1}} H_T(P_1)$

$= \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}$, so does P_2 & P_3

$$\Rightarrow \bigoplus_{i=1}^{3} H^*_T(P_i) \to H^*_T(T^*\mathbb{P}^2) \to \bigoplus_{i=1}^{3} H^*_T(P_i) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

adding $\hbar$ we see $[L_1]+[L_2]|_{P_i} = \begin{cases} (a_3-a_1-\hbar)(a_2-a_1-\hbar) & P_1 \\ -\hbar(a_3-a_2-\hbar) & P_2 \\ -\hbar(a_2-a_3-\hbar) & P_3 \end{cases}$

$\Rightarrow$ Stab. envelopes

(assuming $\mathbb{C}^\times_\hbar \curvearrowright T^*\mathbb{P}^2$ with weight $-\hbar$)