Elementary question

V, vector space $\quad V \cong \bigoplus V_i$ for V_i irreducible w.r.t. some extra data.

E.g.

$G \curvearrowright V$ repr. $\quad V \cong \bigoplus V_i$, V_i irreducible repr.

How about coherent sheaves ?

Similar notion called stability.

$$\begin{array}{c} \mathcal{F} \\ \downarrow \\ (X, \omega) \end{array} \quad X \text{ proj.}$$

Def. $\mathcal{F}$ stable if $\mathcal{F}$ does not cotain any subsheaf E s.t. $\mu(E) > \mu(\mathcal{F})$ $\quad \left(\mu(\mathcal{F}) = \dfrac{\int_X c_1(\mathcal{F}) \wedge \omega^{\dim X - 1}}{rk(\mathcal{F})} \right)$

Q: $\mathcal{F} \cong \bigoplus \mathcal{F}_i$, $\mathcal{F}_i$ stable ?

Ans 1: (DUY / KH)

$$\begin{array}{c} E \\ \downarrow \\ (X, \omega) \end{array} \begin{array}{l} \text{v.b.} \\ \\ \text{Kähler} \end{array}$$

$E \cong \bigoplus E_i$, E_i stable $\Leftrightarrow$ E admits HYM connection

Ans 2: (works for much more objects)

$\forall \mathcal{F}$ pure admits a HN filtration

$0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_{n-1} \subseteq \mathcal{F}_n = \mathcal{F}$ s.t. $\mathcal{F}_i / \mathcal{F}_{i+1}$ is semistable

$\rightsquigarrow$ Works on derived Cat of coherent sheaves.

To mimic Ans 2, we need

(i) full subcats $\mathcal{P}(x)$ for any $x \in \mathbb{R}$ s.t.

$$\mathrm{Hom}_{D^b(X)}(P(x_1), P(x_2)) = 0 \quad \text{if} \quad x_1 < x_2 \quad (P(x_i) \in \mathcal{P}(x))$$

(ii) $\forall\, E \in D^b(X)$, $\exists$ HN filtration, i.e.

$$\begin{tikzcd}[column sep=small]
0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] & \cdots \arrow[r] & E_n \arrow[d] = E \\
& P(\phi_1) \arrow[ul, dashed] & P(\phi_2) \arrow[ul, dashed] & & P(\phi_{n-1})
\end{tikzcd}$$

, $E_i \in D^b(X)$, $\phi_{i+1} > \phi_i$

(iii) $\mathcal{P}(x)[1] = \mathcal{P}(x+1)$

And there is an obvious candidate given by original HN filtration.

① $\forall\, E \in D^b(X)$, we can filter E by cohomological degree, i.e.

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & E^{\le -n} \arrow[r] \arrow[d] & E^{\le -n+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & E^{\le m-1} \arrow[r] & E^{\le m} \arrow[d] = E \arrow[r] & 0 \\
& H^{-n}(E) \arrow[ul, dashed] & H^{-n+1}(E) \arrow[ul, dashed] & H^{-n+2}(E) \arrow[ul, dashed] & & H^{m}(E) \arrow[ul, dashed]
\end{tikzcd}$$

② $H^i(E) \in \mathrm{Coh}(X)$, then we can filter $H^i(E)$ by HN filtration

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & E^i_0 \arrow[r] & E^i_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & E^i_{m_i} = H^i(E) \arrow[r] \arrow[d] & 0 \\
& & HN^1_i(E) \arrow[ul, dashed] & & HN^{m_i}_i(E) \arrow[ul, dashed]
\end{tikzcd}$$

as. $0 \to E^i_j \xrightarrow{\text{inclu.}} E^i_{j+1} \to HN^{j+1}_i(E) \to 0$ is SES.

③ Construct map $E^{\leq -n} \longrightarrow$ "E_0^{-n+1}"

(we don't need to start from $E^{\leq -n}$ as $E^{\leq -n} = H^{-n}(E)$ by definition of truncation)

We got two exact triangles:

$$\begin{tikzcd}
E_i^{-n+1} \arrow[r] \arrow[d] & E^{\leq -n}[1] \arrow[r] \arrow[d, equal] & Q_i^{-n+1}[1] \arrow[d] & \cong \mathrm{Con}(E_i^{-n+1} \to E^{\leq -n}[1]) \\
E_{i+1}^{-n+1} \arrow[r] & E^{\leq -n}[1] \arrow[r] & Q_{i+1}^{-n+1}[1] & \cong \mathrm{Con}(E_{i+1}^{-n+1} \to E^{\leq -n}[1])
\end{tikzcd}$$

And by octahedral axiom:

$$\begin{tikzcd}
E^{\leq -n}[1] \arrow[r] \arrow[dr] & Q_i^{-n+1}[1] \arrow[r] \arrow[d] & E_i^{-n+1}[1] \arrow[d] \\
& Q_{i+1}^{-n+1}[1] \arrow[r] \arrow[dr] & E_{i+1}^{-n+1}[1] \arrow[d] \\
& & \widetilde{Q}^{-n+1,i}[1]
\end{tikzcd}$$

We find comm. diagram

$$\begin{tikzcd}
Q_i^{-n+1}[1] \arrow[r, dash] \arrow[d] & Q_{i+1}^{-n+1}[1] \arrow[r, dash] \arrow[d] & \widetilde{Q}^{-n+1,i}[1] \arrow[d] \\
E_i^{-n+1}[1] \arrow[r, dash] & E_{i+1}^{-n+1}[1] \arrow[r, dash] & HN_{-n+1}^{i+1}(E)[1]
\end{tikzcd}$$

(by uniqueness of Con)

Note that when $i = m_{-n+1}$, $Q_{m_{-n+1}} \cong E^{\leq -n+1}$ (again by uniqueness of Con)

$\Rightarrow E^{\leq -n} \to Q_0^{-n+1} \to Q_1^{-n+1} \to \cdots \to Q_{m_{-n+1}-1}^{-n+1} \to E^{\leq -n+1}$

is our desired map.

④ define suitable map $\mathrm{Coh}(X) \longrightarrow (0,1]$

Again we use the standard slope stability

$$E \longrightarrow \mu(E) = \frac{\deg(E)}{rk(E)}$$ but in a slightly different coordinate

$$\rightsquigarrow E \longrightarrow \deg(E) + i\, rk(E) \in \mathbb{C}$$
$$= |(E)|\, e^{i 2\pi \theta(E)}$$

Fact: $\theta(E) > \theta(F) \Longleftrightarrow \mu(E) > \mu(F)$

And for $\forall\ E, E' \in \mathrm{Coh}(X)$, we have

$$\mathrm{Hom}(E[1], E') = \mathrm{Ext}^{-1}(E, E') = 0$$

However, a big disadvantage of slope stability is that when $\dim_{\mathbb{C}} X \geq 2$, we can not talk about stability of torsion sheave, i.e. pick $E \xrightarrow{\pi} X$ s.t. $rk(E) = 0 = C_1(E)$

Then $\mu(E)$ & $\theta(E)$ are both not defined.

So in additional we need

(iv) $K_0(D^b(X)) = K_0(X) \xrightarrow{Ch} \Lambda$ (some finite rk lattice) $\xrightarrow{\text{additive homo. } Z} \mathbb{C}$

$\forall\ E \in \mathcal{D}(\phi),\ Z(Ch(E)) \in \mathbb{R}_{>0}\, e^{2\pi i \phi}$

And a techical requirement to avoid wild stability condition

(I don't have any understanding on it)

(V)

$$\inf\left\{ \frac{\|Z(\mathrm{Ch}(E))\|}{\|\mathrm{ch}(E)\|} : 0 \neq E \in \mathcal{P}(\phi) \right\} > 0$$

(actually it is a pretty strong property.)

when X is curve, $K_0(X) \xrightarrow{\mathrm{Ch}} \Lambda \cong \mathbb{Z}_{rk} \oplus \mathbb{Z}_{deg}$

$\Rightarrow \|Z(\mathrm{Ch}(E))\| = |rk(E)|^2 + |deg(E)|^2 = \|\mathrm{ch}(E)\|$

$\Rightarrow \inf \frac{\|Z(\mathrm{Ch}(E))\|}{\|\mathrm{ch}(E)\|} = 1 > 0$

$\Rightarrow$ slope stability define a bridgeland stability when X is curve.

Thm. (Toda) When $\dim_{\mathbb{C}}(X) \geq 2$, we can't choose $\mathcal{P}(\phi)$ to be $\mathrm{Coh}(X)$.

Then we need to do some tilling (maybe discuss in the future)

Space of Stability Condition

The space of stability condition $\mathrm{Stab}(X)$ is consisted of $(\mathcal{P}, Z)$ and the topology is defined as follow:

For $\sigma_1 = (\mathcal{P}_1, Z_1)$, $\sigma_2 = (\mathcal{P}_2, Z_2)$

$$|\sigma_1 - \sigma_2| = \sup_{E \in D^b(X),\, E \neq 0} \left\{ |\phi^+_{\sigma_1}(E) - \phi^+_{\sigma_2}(E)|,\ |\phi^-_{\sigma_1}(E) - \phi^-_{\sigma_2}(E)|,\ \|Z_1 - Z_2\|_g \right\}$$

here we filter E by $\mathcal{P}_1$ & $\mathcal{P}_2$

$$\begin{tikzcd}
0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[r] & E_n = E \arrow[dl] \\
P_1(\phi^1_{\sigma_1}) \arrow[u, dashed] & & & P_1(\phi^n_{\sigma_1}) \arrow[u, dashed] &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
0 = E_0' \arrow[r] & E_1' \arrow[d] \arrow[r] & \cdots \arrow[r] & E_{m-1}' \arrow[r] & E_m \arrow[d] = E \\
& P_2(\phi_{\sigma_2}) \arrow[ul, dashed] & & P_2(\phi_{\sigma_2}^m) \arrow[ul, dashed] &
\end{tikzcd}$$

$\phi_{\sigma_1}^+(E) = \phi_{\sigma_1}^n$, $\phi_{\sigma_1}^-(E) = \phi_{\sigma_1}^1$, same for σ_2

And we fix a metric g on $\mathrm{Hom}(\Lambda, \mathbb{C})$.

This is the coarsed topology to make functions

$(\mathcal{P}, Z) \longrightarrow Z$ continuous

& $\forall\, E \in D^b(X)$, $(\mathcal{P}, Z) \longrightarrow \phi^+(E)$

$(\mathcal{P}, Z) \longrightarrow \phi^-(E)$ continuous.

Thm. (Bridgeland)

The map $(\mathcal{P}, Z) \in \mathrm{Stab}(X)$, $(\mathcal{P}, Z) \longrightarrow Z$

is a local homeomorphism, i.e. $\mathrm{Stab}(X)$ admits a cplx mfld

structure with dimension $= \dim_{\mathbb{C}} \Lambda \otimes \mathbb{C}$