building block

(i) two way part

$n \circ\!\!- m \ : T^*\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m)$

(ii) triangle part

$n \ -\!\!\times\!\!- \ m$:

$$\begin{tikzcd} \mathbb{C}^n \arrow[loop above, "B_1"] \arrow[rr, "A"] \arrow[dr, "b"'] & & \mathbb{C}^m \arrow[loop above, "B_2"] \\ & \mathbb{C} \arrow[ur, "a"'] & \end{tikzcd}$$

s.t. ① $B_2A - AB_1 + ab = 0$

② $\nexists\, V \subsetneq \mathbb{C}^m$ s.t. $\mathrm{Im}A + \mathrm{Im}a \subseteq V$ & $B_2(V) \subseteq V$

③ $\nexists\, 0 \neq V \subseteq \mathbb{C}^n$ s.t. $V \subseteq \ker b \cap \ker A$ & $B_1(V) \subseteq V$

(iii) "idenitity"

n — n :

$$\begin{tikzcd} \mathbb{C}^n \arrow[loop above, "B_1"] \arrow[r, "A"] & \mathbb{C}^n \arrow[loop above, "B_2"] \end{tikzcd}$$

s.t. ① $B_2A - AB_1 = 0$

② A inv.

(cylinder)

$\Rightarrow n$ — $n \ \hat{=}\ T^*GL(n)$ (identity in Moore-Tachikawa TQFT)

Prop.

① $n -\!\!\ast\!\!- m \cong GL_m(\mathbb{C}) \times S_{(n, m-n)}$ (when $n<m$)

similar result also holds for $n>m$.

② $n -\!\!\ast\!\!- n \cong T^*GL_n(\mathbb{C}) \times T^*\mathbb{C}^n$

② gluing construction

we define $n_1 -\!\!\ast\!\!- n_2 -\!\!\ast\!\!- n_3$ as

$$n_1 -\!\!\ast\!\!- n_2 \times n_2 -\!\!\ast\!\!- n_3 /\!\!/ GL_{n_2}(\mathbb{C})$$

Similarily define $n_1 -\!\!\ast\!\!- n_2 -\!\!\circ\!\!- n_3$

$$n_1 -\!\!\circ\!\!- n_2 -\!\!\ast\!\!- n_3$$

Fact. $n -\!\!\ast\!\!- m$ is holo. sympl.

$$GL_n(\mathbb{C}) \curvearrowright n -\!\!\ast\!\!- m \curvearrowleft GL_m(\mathbb{C}),$$

two moment maps given by B_1 & B_2

So we can write $n_1 -\!\!\ast\!\!- n_2 -\!\!\ast\!\!- n_3$ as

$$\begin{tikzcd} \mathbb{C}^{n_1} \arrow[loop above, "B_1"] \arrow[rr, "A_1"] \arrow[dr, "b_1"'] & & \mathbb{C}^{n_2} \arrow[loop above, "B_2"] \arrow[rr, "A_2"] \arrow[dr, "b_2"'] & & \mathbb{C}^{n_3} \arrow[loop above, "B_3"] \\ & \mathbb{C} \arrow[ur, "a_1"'] & & \mathbb{C} \arrow[ur, "a_2"'] & \end{tikzcd}$$

Lem. X holo. sympl. $\quad X \times T^*G /\!\!/ G \cong X$ as holo. sympl.

$$\Rightarrow \quad n - \times - m ——— m \;\cong\; n - \times - m$$

$$n - \circ - m ——— m \;\cong\; n - \circ - m$$

Speical cases of bow varieties

(i) $0 - \times - n_1 - \times - n_2 - \times - \cdots - \times - n_m - \times - 0$

$\boxed{\text{affine laumon spaces (handsaw quiver varieties)}}$

i.e. moduli of based rational maps:

$$(\mathbb{P}^1, \infty) \xrightarrow{f} (GL_m(\mathbb{C})/B, \text{ standard flag}) \quad \text{s.t. } \deg f = (n_1, \cdots, n_m)$$

(ii) $\boxed{\text{Type A Nakajima quiver varieties}}$:

$$0 - \circ - n_1 \overset{?}{—} n_2 \overset{?}{—} n_3 \overset{?}{—} \cdots \overset{?}{—} n_m - \circ - 0$$

here ? can be $\times$ or $\circ$, but when $? = \times$, the n_i, n_{i+1} in two edges must be equal.

heuristic: $n - \times - n \cong T^*GL_n(\mathbb{C}) \times T^*\mathbb{C}^n$

$$\Rightarrow \quad n - \times - n /\!/ GL_n(\mathbb{C}) \cong T^*\mathbb{C}^n = \begin{array}{c} \boxed{1} \\ | \\ \boxed{n} \end{array} \quad \text{(framing)}$$

in general $\underbrace{n - \times - n - \times - \cdots - \times - n}_{\#\text{“}-\times-\text{”} = m} \cong (T^*GL_n(\mathbb{C}))^m \times (T^*\mathbb{C}^n)^m /\!/ (GL_n(\mathbb{C}))^m$

$$\cong (T^*\mathbb{C}^n)^m = \begin{array}{c} \boxed{m} \\ | \\ \boxed{n} \end{array}$$

(iii) $\boxed{\text{Some Coulomb branches}}$

$$0 - \times - n_1 \overset{?}{—} n_2 \overset{?}{—} n_3 \overset{?}{—} \cdots \overset{?}{—} n_m - \times - 0$$

similar to (ii), but when $? = \ominus$, the n_i, n_{i+1} in two edges must be equal.

Obviously (i) $\subseteq$ (iii)

Prop. $\forall\ X \in$ (iii), up to codim 2, X admits a cover of (i)

E.g.

$$X = 0 \overset{}{-\!\times\!-} n_1 -\!\ominus\!- n_1 -\!\times\!- n_2 -\!\ominus\!- n_2 -\!\times\!- 0$$

$$= 0 -\!\times\!- n_1 \underset{B_1}{\overset{A_1}{\rightleftarrows}} n_1 -\!\times\!- n_2 \underset{B_2}{\overset{A_2}{\rightleftarrows}} n_2 -\!\times\!- 0$$

$A_1, B_1 \in \mathfrak{gl}(n_1)$; $A_2, B_2 \in \mathfrak{gl}(n_2)$

if one of A_1, B_1 & one of A_2, B_2 invertible

$\Rightarrow$ $(A_1, B_1) \in T^*GL(n_1)$, $(A_2, B_2) \in T^*GL(n_2)$

$\Rightarrow$ belong to some $X = 0 -\!\times\!- n_1 -\!-\!- n_2 -\!\times\!- n_2 -\!-\!- n_2 -\!\times\!- 0$

$$= 0 -\!\times\!- n_1 -\!\times\!- n_2 -\!\times\!- 0$$

Combinatoric 3d Mirror Symmetry

$\forall$ X bow variety of the form

$0 \overset{?}{-} n_1 \overset{?}{-} n_2 \overset{?}{-} \cdots \overset{?}{-} n_m \overset{?}{-} 0$ (☆)

we change $-\!\times\!-$ to $-\!\ominus\!-$, $-\!\ominus\!-$ to $-\!\times\!-$, $—$ to $—$,

Then we get dual bow variety $X^!$, and

$$X \overset{3d}{\longleftrightarrow} X^!$$

Hanany – Witten transition

$$\cdots\ n_1 \;-\!\ominus\!-\; n_2 \;-\!\times\!-\; n_3 \ \cdots$$

$$\updownarrow$$

$$\cdots\ n_1 \;-\!\times\!-\; n_2+1-n_1-n_3 \;-\!\ominus\!-\; n_3 \ \cdots$$

these two guys are holo. sympl. iso.

Charge of brane

$\forall$ bow variety of the form (☆), we can define charge of $-\!\times\!-$ in $n_i -\!\times\!- n_{i+1}$ & charge of $-\!\ominus\!-$ in $n_i -\!\ominus\!- n_{i+1}$ as following.

charge of $-\!\times\!-$ in $n_i -\!\times\!- n_{i+1}$ $= n_{i+1} - n_i + \#\, -\!\ominus\!-$ on the left.

charge of $-\!\ominus\!-$ in $n_i -\!\ominus\!- n_{i+1}$ $= n_i - n_{i+1} + \#\, -\!\times\!-$ on the right.

Fact: charge is inv. under HW transition.

Prop: Botta – Rimányi

given bow variety X of the form (☆), if charge of $\forall$ $-\!\times\!-$ & $-\!\ominus\!-$ >0, then X can be HW move to the following standard form.

$$0 -\!\times\!- n_1 -\!\times\!- n_2 -\!\times\!- \cdots -\!\times\!- n_m -\!\ominus\!- n_{m+1} -\!\ominus\!- n_{m+2} -\!\ominus\!- \cdots -\!\ominus\!- n_{m+k} -\!\ominus\!- 0 \quad (\#)$$

s.t. $0 < n_1 < n_2 < \cdots < n_m > n_{m+1} > \cdots > n_{m+k} > 0$

it's easy to see (#) $\cong T^*G/P \times_{\mathfrak{g}^*} S_{(n_m-n_{m+1},\, n_{m+1}-n_{m+2},\, \cdots,\, n_{m+k})}$

here the partition of P is given by $(n_1, n_2-n_1, \cdots, n_m-n_{m-1})$

This Prop. can be understood as following:

finite type A Nakajima quiver varieties.

We know 3d Mirror of S_3 variety $\mathcal{N}_\lambda \cap S_{\lambda'}$

is $\mathcal{N}_{\lambda'} \cap S_\lambda$, also a S_3 variety.

here $\mathcal{N}_\lambda$, $\mathcal{N}_{\lambda'}$ are nilpotent orbit closures, S_λ, $S_{\lambda'}$ are slodowy slices.

The key ingredient of this mirror phenomenon is

nilpotent orbits closure $\xleftrightarrow[1:1]{}$ Slodowy slices

However, if we resolve $\mathcal{N}_\lambda$, we lost such 1 to 1 correspondence, as a sympl. singularity may admits many sympl. resolutions. So we need to extend to notion of slodowy slices.

Matrices free description of such extention:

Slodowy slices $\longleftrightarrow$ sl_2 repr. on $\mathfrak{g}$

extended Slodowy slices $\longleftrightarrow$ sl_2 repr. (e_1, f_1, h_1) on $\mathfrak{g}$;

sl_2 repr. (e_2, f_2, h_2) on $\mathfrak{g}'$, $\mathfrak{g}'$ = lie algebra of id component of $Z_G(e_2, f_2, h_2)$;

⋮